

Null Frenet-Serret Dynamics

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(Dated: February 7, 2008)

We consider the Frenet-Serret geometry of null curves in a three and a four-dimensional Minkowski background. We develop a theory of deformations adapted to the Frenet-Serret frame. We exploit it to provide a Lagrangian description of the dynamics of geometric models for null curves.

PACS numbers: 02.40.Hw, 11.30.-j, 14.80.-j

Dedicated to Mike Ryan on his sixtieth birthday.

The notion of a relativistic point-like object, or particle, is an idealization that has guided the development of reparametrization invariant theories, in particular, string theory and its membrane descendants [1, 2]; it has also informed our understanding of general relativity as a dynamical theory [3, 4]. The massive relativistic particle, with an action proportional to the length of its worldline, represents the simplest global geometrical quantity invariant under reparametrizations. The particle follows a geodesic of the ambient spacetime – the worldline curvature vanishes. A natural extension is to consider higher order geometrical models for particles, described by an action that depends on the curvatures of the worldline. While the initial motivation for their introduction was their value as toy models for higher dimensional relativistic systems such as strings [5], it has turned out that they possess interesting features in their own right; they model spinning particles [6, 7, 8] and they are relevant to various integrable systems [9]. The approach, traditionally adopted, is to go Hamiltonian, with an eye on canonical quantization. The disadvantage has been that models have been examined on a case by case basis with a tendency to lose sight of shared features (see however [10] for a way to remedy this shortcoming). An alternative Lagrangian approach, developed in Ref. [11], describes the dynamics of these higher order geometric models for particles in terms of the Frenet-Serret representation of the worldline in Minkowski spacetime. This representation exploits the existence of a preferred parametrization for curves – parametrization by arclength. A clear advantage of this approach is that the conserved quantities associated with the underlying Poincaré symmetry are described directly in terms of the *geometrically signifi-*

cant worldline curvatures.

In these geometrical models for relativistic particles, the worldline is timelike, so that spacelike normal vector fields can be consistently defined. As it stands, therefore, this framework does not admit null curves, where arc-length vanishes and the normal is tangential to the curve. Besides their intrinsic interest as essentially relativistic objects, however, null curves are also potentially valuable in the construction of geometric models for light-like relativistic extended objects [12, 13, 14]. An extension of the Frenet-Serret representation to the case of null curves was constructed recently [15], and applied to the simplest geometrical model based on an action proportional to pseudo arclength involving second derivatives (see also Refs. [16] for interesting work on the geometry of null curves). Subsequently, in the special case of a particle moving in $2+1$ dimensions, both the model proportional to pseudo arclength [17], and a model depending on the first pseudo-curvature have been considered [18, 19].

In this paper, we develop a theory of deformations of the geometry of null curves adapted to a Frenet-Serret frame. We work in four spacetime dimensions. The first variations of the geometrical action is described directly in terms of curvatures. In the case of the models previously considered, the description of the dynamics is simplified considerably. This streamlining makes it feasible to consider interesting four-dimensional generalizations. We consider, in particular, a model linear in the first curvature in a four-dimensional background. We show that its dynamics can be framed in terms of the dynamics of a fictitious non-relativistic particle moving in two dimensions. We also consider a model linear in the second curvature in a four-dimensional background.

We begin by briefly summarizing the Frenet-Serret geometry for null curves as given in Ref. [15], to which we refer the interested reader for a more detailed treatment. A curve in four-dimensional Minkowski spacetime is described by the embedding

$$x^\mu = X^\mu(\lambda), \quad (1)$$

with λ an arbitrary parameter ($\mu, \nu, \dots = 0, 1, 2, 3$); x^μ are local coordinates in Minkowski spacetime, and X^μ are

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the embedding functions. The tangent vector is $\dot{X}^\mu = dX^\mu/d\lambda$ (we denote with an overdot a derivative with respect to λ). The curve is null, it lives on the light cone,

$$\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = \dot{X} \cdot \dot{X} = 0, \quad (2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We use a signature $(+ - - -)$ throughout the paper.

The infinitesimal pseudo arclength for a null curve can be defined as

$$d\sigma = \left(-\ddot{X} \cdot \ddot{X}\right)^{1/4} d\lambda. \quad (3)$$

This is a natural generalization of the arc-length for non-null curves $d\sigma = (-\dot{X} \cdot \dot{X})^{1/2}$; arc-length itself clearly will not do for null curves since it vanishes. We denote with a prime derivation with respect to σ . The pseudo arclength, like arclength, is invariant under reparametrizations.

The Frenet-Serret frame adapted to this class of curves is given by the four spacetime vectors $\{e_+, e_-, e_1, e_2\}$, where

$$\begin{aligned} e_+ &= X', \\ e_1 &= e_+' , \\ e_+^2 &= e_-^2 = 0, \\ e_+ \cdot e_1 &= e_+ \cdot e_2 = e_- \cdot e_1 = e_- \cdot e_2 = e_1 \cdot e_2 = 0, \\ e_+ \cdot e_- &= -e_1 \cdot e_1 = -e_2 \cdot e_2 = 1. \end{aligned}$$

Two of the vectors are null, and two are spacelike. We assume that the curve is sufficiently smooth so that this frame is well defined.

The Frenet-Serret equations for a null curve are

$$e_+' = e_1, \quad (4)$$

$$e_1' = \kappa_1 e_+ + e_-, \quad (5)$$

$$e_-' = \kappa_1 e_1 + \kappa_2 e_2, \quad (6)$$

$$e_2' = \kappa_2 e_+, \quad (7)$$

where the two curvatures are given by

$$\kappa_1 = \frac{1}{2} X''' \cdot X''', \quad (8)$$

$$\kappa_2 = \sqrt{-X'''' \cdot X'''' - (X''' \cdot X''')^2}. \quad (9)$$

The null curve is characterized by two curvatures, whereas a non-null curve is described by three curvatures, in a four-dimensional background. The difference is due to the fact that the null curve is constrained to a light cone. In a way, κ_1 can be thought as the analogue of the second Frenet-Serret curvature (or torsion) of the non-null case, in the sense that κ_1 depends on three derivatives with respect to pseudo arclength. Note that null curves with constant curvatures κ_1, κ_2 are helices on the light cone [16]. If $\kappa_2 = 0$, the curves lives in a 2+1 dimensional Minkowski space.

We consider reparametrization invariant models for null curves whose dynamics is determined by an action of the form

$$S[X] = \int d\sigma L(\kappa_1, \kappa_2), \quad (10)$$

where the Lagrangian L is an arbitrary function of the two curvatures κ_1, κ_2 as given by Eqs. (8), (9). The first variation of this action will yield both the equations of motion and the Noether charge. The latter gives the conserved quantities associated with the underlying Poincaré symmetry, linear and angular momentum (see *e.g.* [11]). In the first variation of the action, we consider only infinitesimal deformations that maintain the null character of the curve.

We make an infinitesimal deformation of the curve

$$X \rightarrow X + \delta X. \quad (11)$$

We can expand the deformation with respect to the Frenet-Serret frame as

$$\delta X = \epsilon_+ e_+ + \epsilon_- e_- + \epsilon_1 e_1 + \epsilon_2 e_2. \quad (12)$$

This is always a convenient strategy when one is interested in the variation of reparametrization independent quantities. This is because the deformation along e_+ is an infinitesimal reparametrization of the curve, so that, setting $\delta_{\parallel} X = \epsilon_+ e_+$, for the infinitesimal deformation of the infinitesimal pseudo arclength (3) we have

$$\delta_{\parallel} d\sigma = \epsilon_+' d\sigma. \quad (13)$$

For any worldline scalar function $f(X)$, its parallel deformation is given by

$$\delta_{\parallel} f = \epsilon_+ f'. \quad (14)$$

Therefore, for the reparametrization invariant geometrical model defined by the action (10), we have

$$\delta_{\parallel} S = \int d\sigma (\epsilon_+ L)'. \quad (15)$$

The deformation along e_+ contributes only a boundary term. The non-trivial part of the deformation is given by the remainder, which we denote by

$$\delta_{\perp} X = \epsilon_- e_- + \epsilon_1 e_1 + \epsilon_2 e_2. \quad (16)$$

As we are interested only in deformations that preserve the null character of the curve, we consider

$$\begin{aligned} \dot{X} \cdot \delta_{\perp} \dot{X} &= \dot{X} \cdot \frac{d}{d\lambda} (\delta_{\perp} X) \\ &= \left(\frac{d\sigma}{d\lambda}\right)^2 e_+ \cdot (\epsilon_- e_- + \epsilon_1 e_1 + \epsilon_2 e_2)' \\ &= \left(\frac{d\sigma}{d\lambda}\right)^2 (\epsilon_1 + \epsilon_-'). \end{aligned} \quad (17)$$

The condition $\delta_\perp(\dot{X} \cdot \dot{X}) = 0$ implies the constraint

$$\epsilon_1 + \epsilon_- = 0 \quad (18)$$

on the components of the deformation. Thus deformations that preserve null curves are completely specified by two independent normal variations, ϵ_1, ϵ_2 . However, in order to keep the deformation of the geometry local in its components, it is convenient to take ϵ_-, ϵ_2 as the independent variations. In the case of a three-dimensional Minkowski background, there is only one independent component of the normal deformation, which we take to be ϵ_- .

For the variation of the pseudo arclength (3), a straightforward calculation gives, using (18),

$$\delta_\perp d\sigma = \frac{d\sigma}{2}(-\epsilon_-''' + \kappa_1' \epsilon_- + \kappa_2 \epsilon_2) = \Omega d\sigma, \quad (19)$$

where we have defined the quantity Ω for later convenience. For any worldline scalar f it follows that

$$\delta_\perp f' = -\Omega f' + (\delta_\perp f)'. \quad (20)$$

Applying this relation to the spacetime vector e_+ , we obtain

$$\begin{aligned} \delta_\perp e_+ &= \frac{1}{2}(\epsilon_-''' - 2\kappa_1 \epsilon_-' - \kappa_1' \epsilon_- + \kappa_2 \epsilon_2) e_+ \\ &+ (-\epsilon_-'' + \kappa_1 \epsilon_-) e_1 + (\epsilon_2' + \kappa_2 \epsilon_-) e_2, \end{aligned} \quad (21)$$

and for the vector e_1 a similar computation produces

$$\begin{aligned} \delta_\perp e_1 &= \frac{1}{2}[\epsilon_-'''' - \kappa_1'' \epsilon_- - 3\kappa_1' \epsilon_-' - 4\kappa_1 \epsilon_-'' \\ &+ 2(\kappa_1^2 + \kappa_2^2) \epsilon_- + \kappa_2' \epsilon_2 + 3\kappa_2 \epsilon_2'] e_+ \\ &+ (-\epsilon_-'' + \kappa_1 \epsilon_-) e_- + (\epsilon_2' + \kappa_2 \epsilon_-)' e_2. \end{aligned} \quad (22)$$

The variations of the other two frame vectors e_-, e_2 can be calculated along the same lines, but we will not need them in the following.

We are now in a position to derive the deformation of any geometrical quantity associated with the curve. Let us consider the variation of the first curvature κ_1 . From Eq. (5), we have

$$\delta_\perp(e_1') = (\delta_\perp \kappa_1) e_+ + \kappa_1 \delta_\perp e_+ + \delta_\perp e_-, \quad (23)$$

and dotting with e_- , we obtain

$$\delta_\perp \kappa_1 = e_- \cdot \delta_\perp(e_1') - \kappa_1 e_- \cdot \delta_\perp e_+. \quad (24)$$

We can read off the second term from Eq. (21). For the first term, note that

$$\begin{aligned} e_- \cdot \delta_\perp(e_1') &= -\Omega e_- \cdot e_1' + e_- \cdot (\delta_\perp e_1)' \\ &= -\Omega \kappa_1 + (e_- \cdot \delta_\perp e_1)' - e_-' \cdot \delta_\perp e_1 \\ &= -\Omega \kappa_1 + (e_- \cdot \delta_\perp e_1)' - \kappa_2 e_2 \cdot \delta_\perp e_1. \end{aligned}$$

Substituting these expressions in Eq. (24), and using Eqs. (21), (22), we obtain

$$\begin{aligned} \delta_\perp \kappa_1 &= \frac{1}{2}[\epsilon_-'''' - \kappa_1'' \epsilon_- - 3\kappa_1' \epsilon_-' - 4\kappa_1 \epsilon_-'' \\ &+ 2(\kappa_1^2 + \kappa_2^2) \epsilon_- + \kappa_2' \epsilon_2 + 3\kappa_2 \epsilon_2']' \\ &+ \kappa_2 (\epsilon_2' + \kappa_2 \epsilon_-)' + \kappa_1 (\kappa_1 \epsilon_-' - \kappa_2 \epsilon_2). \end{aligned} \quad (25)$$

An analogous calculation gives that the variation of the second curvature is

$$\begin{aligned} \delta_\perp \kappa_2 &= [(\epsilon_2' + \kappa_2 \epsilon_-)'' - \kappa_1 \epsilon_2' - \kappa_2 \epsilon_-'']' - \kappa_2^2 \epsilon_2 \\ &+ \kappa_2 \kappa_1 \epsilon_-' - \kappa_1 (\epsilon_2' + \kappa_2 \epsilon_-)'. \end{aligned} \quad (26)$$

Note that in both expressions a large part of the variation is in the form of a total derivative.

The expressions we have derived allow us to obtain the variation of any geometric quantity associated with the curve. In particular, one can consider geometric models defined by an action of the form (10). The simplest such model is proportional to pseudo arclength [15, 17]

$$S[X] = 2\alpha \int d\sigma. \quad (27)$$

Using Eqs. (15), (19), its variation is found to be

$$\delta S = \alpha \int d\sigma (\kappa_1' \epsilon_- + \kappa_2 \epsilon_2) + \alpha \int d\sigma (-\epsilon_-'' + 2\epsilon_+)' . \quad (28)$$

One can immediately read off the equations of motion to be

$$\kappa_1 = \text{const.}, \quad \kappa_2 = 0. \quad (29)$$

The solutions are null helices constrained to a 2+1 dimensional linear subspace of the Minkowski spacetime. The total derivative in the variation of the action, using standard techniques (see *e.g.* [11]), gives the conserved linear and angular momentum associated with the underlying Poincaré symmetry,

$$P = \alpha (e_- - \kappa_1 e_+), \quad (30)$$

$$M^{\mu\nu} = P^{[\mu} X^{\nu]} + \alpha e_+^{[\mu} e_1^{\nu]}. \quad (31)$$

Note that the linear momentum is along the null vectors e_+, e_- . In this sense we can consider it as *tangential* to the curve. The conserved mass, or first Casimir of the Poincaré group, is

$$M^2 = P^2 = -2\alpha^2 \kappa_1. \quad (32)$$

In order to have a positive mass, it is necessary that $\kappa_1 < 0$, which implies that e_1' is spacelike, as follows from Eq. (8). Therefore the constant value of κ_1 is related to the Casimirs of the underlying Poincaré symmetry. The Pauli-Lubanski pseudo-vector is

$$S_\mu = \frac{1}{2} \frac{1}{\sqrt{|M^2|}} \varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} = -\frac{\alpha^2}{2\sqrt{|M^2|}} e_{\mu 2}, \quad (33)$$

with $\varepsilon_{\mu\nu\rho\sigma}$ the Levi-Civita tensor density, and we use the convention $\varepsilon_{\mu\nu\rho\sigma}e^\mu_+e^\nu_-e^\rho_1e^\sigma_2 = +1$. The spin pseudo-vector is spacelike. The second Poincaré Casimir is then

$$|M^2|S^2 = -\frac{\alpha^4}{4}. \quad (34)$$

Moreover, we have

$$S^2 = -(1/8)\alpha^2\kappa_1^{-1}. \quad (35)$$

If we consider a 2+1 ambient Minkowski spacetime, besides $\kappa_2 = 0$, the only change is in the definition of the spin pseudo-vector. We have

$$J_\mu = \varepsilon_{\mu\rho\sigma}M^{\rho\sigma} = \varepsilon_{\mu\rho\sigma}P^\rho X^\sigma - \alpha e_\mu +, \quad (36)$$

where now we use the convention $\varepsilon_{\mu\nu\rho}e^\mu_+e^\nu_-e^\rho_1 = +1$. Note that the non-orbital part of J_μ is tangential. It follows that the second Casimir takes the form [17]

$$S = J_\mu P^\mu = -\alpha^2. \quad (37)$$

Let us consider now a model that involves the first curvature. The simplest one is linear in κ_1 [18, 19],

$$S = 2 \int d\sigma (\alpha + \beta\kappa_1). \quad (38)$$

In the simpler case of a 2+1 dimensional Minkowski background, using Eqs. (19), (25), we find that the model gives the equation of motion

$$\beta\kappa_1''' - \frac{3\beta}{2}(\kappa_1^2)' + \alpha\kappa_1' = 0, \quad (39)$$

which can be integrated twice to give

$$\frac{1}{2}\beta\kappa_1'^2 - \frac{\beta}{2}\kappa_1^3 + \frac{1}{2}\alpha\kappa_1^2 - \gamma_{(3)}\kappa_1 = E_{(3)}, \quad (40)$$

where $\gamma_{(3)}$ and $E_{(3)}$ are constants that can be expressed in terms of the Casimirs for this system. At the level of the curvatures, the dynamics is described by the motion of a fictitious particle moving in one dimension in a cubic potential. The system is clearly integrable by quadratures: κ_1 can be expressed in terms of elliptic integrals [18, 19]. It is clear from Eq.(40) that there are solutions with bounded periodic κ_1 . To obtain the corresponding trajectories requires one to integrate the curvature.

The linear and angular momentum are given by

$$\begin{aligned} P &= (-\beta\kappa_1'' + \beta\kappa_1^2 - \alpha\kappa_1)e_+ + \beta\kappa_1'e_1 \\ &\quad + (\alpha - \beta\kappa_1)e_-, \\ M^{\mu\nu} &= P^{[\mu}X^{\nu]} + 2\beta e^{[\mu}_-e^{\nu]}_1 + (\alpha + \beta\kappa_1)e^{[\mu}_+e^{\nu]}_1. \end{aligned} \quad (41)$$

$$(42)$$

Note that the linear momentum acquires a term in the normal direction e_1 . This is analogous to what happen to curvature-dependent models for non-null curves (see e.g. [11]). The spin pseudo-vector takes the form

$$J_\mu = \varepsilon_{\mu\rho\sigma}P^\rho X^\sigma + 2\beta e_{\mu-} - (\alpha + \beta\kappa_1)e_{\mu+}. \quad (43)$$

The two Casimirs are therefore

$$\begin{aligned} M^2 &= 2(-\beta\kappa_1'' + \beta\kappa_1^2 - \alpha\kappa_1)(\alpha - \beta\kappa_1) \\ &\quad - \beta^2(\kappa_1')^2 \end{aligned} \quad (44)$$

$$S = -2\beta\left(\beta\kappa_1'' - \frac{3\beta}{2}\kappa_1^2 + \alpha\kappa_1\right) - \alpha^2. \quad (45)$$

The latter expression identifies the constant $\gamma_{(3)}$ as

$$\gamma_{(3)} = -(1/2\beta)(S + \alpha^2). \quad (46)$$

We reproduce Eq.(40) subtracting the Casimirs to eliminate κ_1'' :

$$\beta^2\kappa_1'^2 - [\beta^{-1}(S + \alpha^2) - \beta\kappa_1^2](\alpha - \beta\kappa_1) = -M^2, \quad (47)$$

which identifies the constant $E_{(3)}$ that appears in Eq. (40) as

$$E_{(3)} = \frac{1}{2\beta^2}(S + \alpha^2 - \beta M^2). \quad (48)$$

We extend now our consideration of this model to a 3+1 dimensional background. The variation of the action (38) gives the two equations of motion

$$\beta\kappa_1''' - \frac{3\beta}{2}(\kappa_1^2)' - \beta(\kappa_2^2)' + \alpha\kappa_1' = 0, \quad (49)$$

$$2\beta\kappa_2'' - \beta\kappa_1\kappa_2 + \alpha\kappa_2 = 0. \quad (50)$$

The first equation again possesses a first integral,

$$\beta\kappa_1'' - \frac{3\beta}{2}\kappa_1^2 - \beta\kappa_2^2 + \alpha\kappa_1 = \gamma_{(4)}, \quad (51)$$

where $\gamma_{(4)}$ is another constant. We have therefore two coupled differential equations of second order. Unlike the 2+1 case, the presence of κ_2 stymies the second integration of Eq.(51). However, it is clear from Eqs.(50) and (51) that they can be derived from a potential: we have

$$\frac{1}{2}\beta(\kappa_1')^2 + 2\beta(\kappa_2')^2 + V(\kappa_1, \kappa_2) = E_{(4)}, \quad (52)$$

where

$$V(\kappa_1, \kappa_2) = -\frac{1}{2}\beta\kappa_1^3 + \frac{1}{2}\alpha\kappa_1^2 - (\gamma_{(4)} + \beta\kappa_2^2)\kappa_1 + \alpha\kappa_2^2, \quad (53)$$

and $E_{(4)}$ is another constant. The dynamics is described by the motion of a fictitious particle moving two dimensions.

The linear momentum is changed by the addition of a term in the direction e_2 ,

$$\begin{aligned} P &= (-\beta\kappa_1'' + \beta\kappa_1^2 - \alpha\kappa_1)e_+ + (\alpha - \beta\kappa_1)e_- \\ &\quad + \beta\kappa_1'e_1 + 2\beta\kappa_2'e_2, \end{aligned} \quad (54)$$

so that

$$\begin{aligned} M^2 &= 2(-\beta\kappa_1'' + \beta\kappa_1^2 - \alpha\kappa_1)(\alpha - \beta\kappa_1) \\ &\quad - \beta^2(\kappa_1')^2 - 4\beta^2(\kappa_2')^2. \end{aligned} \quad (55)$$

The conserved angular momentum is modified to

$$M^{\mu\nu} = P^{[\mu} X^{\nu]} + 2\beta e^{[\mu} e^{\nu]}_1 + (\alpha + \beta\kappa_1) e^{[\mu} e^{\nu]}_1 + 2\beta\kappa_2 e^{[\mu} e^{\nu]}_2, \quad (56)$$

and the spin pseudo-vector takes the form

$$2\sqrt{|M^2|}S = 2\beta(\beta\kappa_2\kappa_1' - \beta\kappa_1\kappa_2' - \alpha\kappa_2')e_+ + 4\beta^2\kappa_2'e_- + 2\beta\kappa_2(\alpha - \beta\kappa_1)e_1 + (-2\beta^2\kappa_1'' + 3\beta^2\kappa_1'^2 - 2\alpha\beta\kappa_1 - \alpha^2)e_2. \quad (57)$$

Now using the conservation law (51), we have

$$M^2 S^2 = 2\beta^3\kappa_2'(\beta\kappa_2\kappa_1' - \beta\kappa_1\kappa_2' - \alpha\kappa_2') - \beta^2\kappa_2^2(\alpha - \beta\kappa_1)^2 - \frac{1}{4}(\alpha^2 + 2\beta\gamma_{(4)})^2, \quad (58)$$

together with

$$M^2 = -2\left(\gamma_{(4)} + \beta\kappa_2^2 + \frac{\beta}{2}\kappa_1^2\right)(\alpha - \beta\kappa_1) - \beta^2(\kappa_1')^2 - 4\beta^2(\kappa_2')^2. \quad (59)$$

The latter reproduces Eq.(52) with the identification

$$2\beta E_{(4)} = -M^2 - 2\alpha\gamma_{(4)}. \quad (60)$$

There are two first order equations for κ_1 and κ_2 . which suggests that the system is integrable, We are unable, however, to find an explicit reduction.

Finally, we comment briefly on a model linear in the second curvature

$$S[X] = 2\lambda \int d\sigma \kappa_2. \quad (61)$$

Using Eqs. (19) and (26), the variation of the action (61) gives the corresponding equations of motion

$$\lambda\kappa_2''' - 2\lambda\kappa_1\kappa_2' + \lambda\kappa_1'\kappa_2 = 0, \quad (62)$$

$$2\lambda\kappa_1'' + \lambda\kappa_2^2 = 0. \quad (63)$$

These equations can be decoupled. One solves Eq.(62) for κ_1 :

$$\kappa_1 = -\kappa_2^2 \int d\sigma \kappa_2''' / \kappa_2^3, \quad (64)$$

and substitutes into Eq.(63), which gives a fifth order equation for κ_2 alone.

To summarize, the Frenet-Serret frame provides a natural description of a null curve. We have shown how the deformations of the curve can be described in a way which is adapted to the frame; in particular, we have considered deformations that preserve the null character of the curve and obtained explicit expressions for the deformations of the curvatures. These curvatures are used to construct geometrical models for null curves. We have examined the first variation of several simple actions, demonstrating that the corresponding Euler-Lagrange equations can be cast as a set of coupled non-linear ODEs for the curvatures.

Acknowledgements

We acknowledge partial support from CONACyT under grants 44974-F, C01-41639 and PROMEP-2003.

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